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Lucas Graceful Labeling for T_P–Tree

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Abstract

Let G be a (p,q)-graph. An injective function $f: V(G) \to \{l_0, l_1, l_2, l_3 . . . l_a\}, (a \in N)$, is said to be Lucas graceful labeling if an induced edge labeling $f_1(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{l_1, l_2, l_3 . . . l_q\}$. Then G is called Lucas graceful graph if it admits Lucas graceful labeling. In this paper, we show that the Transformed Tree (T_P-Tree) is Lucas graceful graph.

Keywords: Graceful labeling, Lucas Graceful labeling and Lucas graceful graph.

Introduction

By a graph, we mean a finite undirected graph without loops or multiple edges. A tree is a connected graph T without cycles. The graph labeling involves labeling vertices or edges or both, using integers subject to certain conditions. The concept of graceful labeling was introduced by Rosa [4] in 1967. A complete and current summary of graceful and non-graceful results along with some unproven conjectures can be found in Gallian [2] dynamic survey of graceful labeling. M.A.Perumal, S,Navaneetha krishnan and A.Nagarajan [3] have proved that some graphs admit Lucas graceful labeling.

A function f is a graceful labeling of a graph G with q edges if f is an injection from the vertices of G to the set {0, 1, 2, 3. . .q} such that when each edge uv is assigned the label |f(u)- f(v)|, the resulting edge labels are distinct. A graph is graceful if it has a graceful labeling.

Let G be a (p,q) - graph. An injective function $f : V(G) \to \{l_0, l_1, l_2, l_3 . . . l_a\}, (a \in N)$, is said to be Lucas graceful labeling if an induced edge labeling $f_1(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{l_1, l_2, l_3 . . . l_q\}$ with the assumption of $l_0 = 0, l_1 = 1, l_2 = 3, l_3 = 4, l_4 = 7, l_5 = 11, l_6 = 18 . . .$ Then G is called Lucas graceful graph if it admits Lucas graceful labeling. In this paper, we show that the Transformed Tree (T_P-Tree) is Lucas graceful graph.

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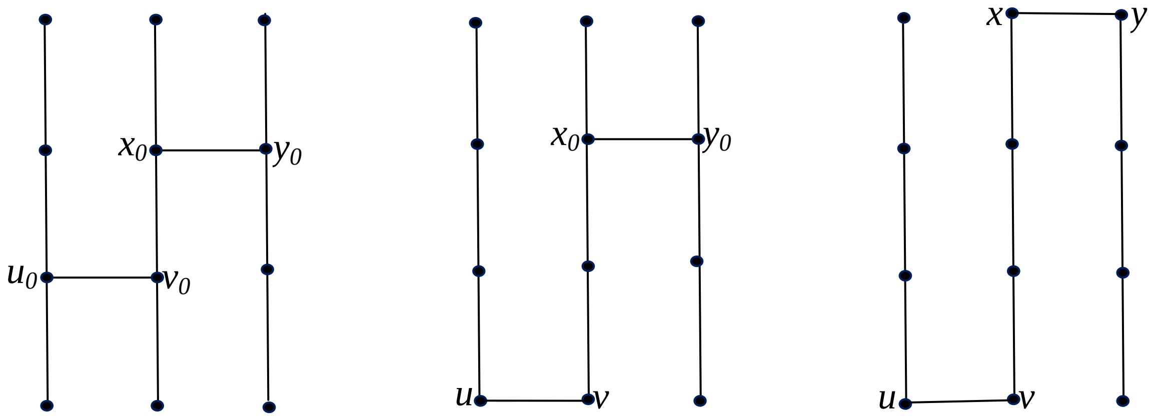
Main results

Definition 2.1 - T_P- Tree (Transformed Tree):

Let T be a tree and u_0 and v_o be two adjacent vertices in T. Let there be two pendent vertices u and v in T such that the length of u_0-u path is equal to the length of v_0-v path. If the edge $u_0 v_0$ is deleted from T and u and v are joined by an edge uv, then such a transformation of T is called elementary parallel transformation (or an ept) and the edge $u_0 v_0$ is called transformable edge. If by a sequence of ept's T can be reduced to a path then T is called a T_P-Tree (transformed Tree) and any such sequence regarded as a composition of mappings (ept's) denoted by P, is called parallel transformation of T. The path, the image of T under P is denoted by P(T). A T_p-tree and a sequence of two ept's reducing it to a path are illustrated in Fig. 1.



a) A T_p-tree T b) An ept $P_1(T)$ c) Second ept $P_2(T)$

Fig. 1

Theorem : 2.2.

Transformed tree (T_P-Tree) G with 4n vertices is a Lucas graceful graph.

Proof : Let G be a graph with vertex set V(G) = { u_1, u_2….u_n } ⋃ { v_1, v_2,….v_n } ⋃ { w_1,w_2……w_n } ⋃ { x_1,x_2…..x_n } V (G) | = 4n, | E (G) | = 4n-1

The arbitrary labeling of T_p-tree G is shown in figure.

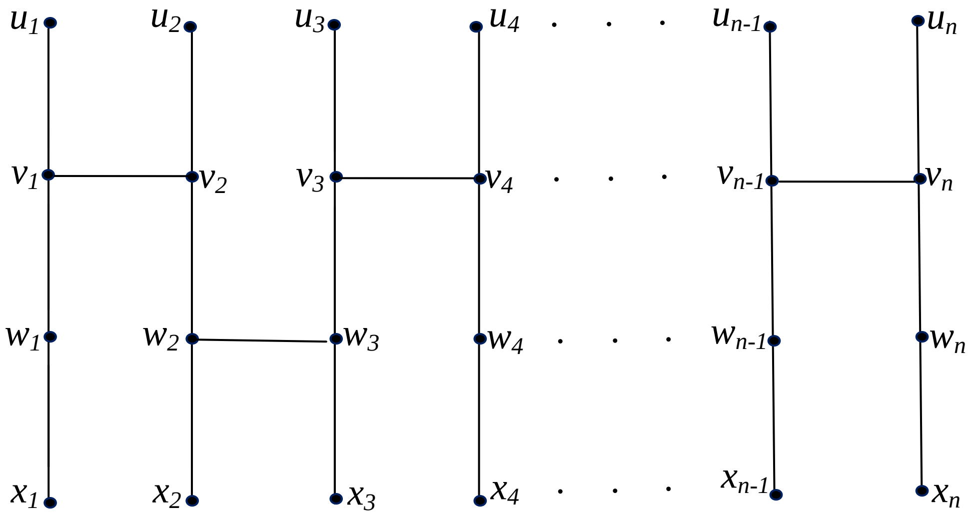


Fig. 2 :Arbitrary labeling of T_p-tree G

The mapping $f : V(G) \rightarrow \{l_0, l_1, ..., l_a\}, a \in N$ is defined by,

$f(u_1) = l_1$, $f(v_1) = l_0$, $f(w_1) = l_2$, $f(x_1) = l_4$

Case (i) when n is even, the corresponding labeling is defined by,

$f(u_{2i}) = l_{4n-2-8(i-1)}, 1 \le i \le n-3,\ f(v_{2i}) = l_{4n-1-8(i-1)}, 1 \le i \le n-3,$

$f(w_{2i}) = l_{4n-3-8(i-1)}, 1 \le i \le n-3,\ f(x_{2i}) = l_{4n-4-8(i-1)}, 1 \le i \le n-3,$

$f(x_{2n-6}) = l_3, 1 \le i \le n-3,\ f(u_{2i+1}) = l_{4n-8-8(i-1)}, 1 \le i \le n-4$

$f(v_{2i+1}) = l_{4n-7-8(i-1)}, 1 \le i \le n-4,\ f(w_{2i+1}) = l_{4n-5-8(i-1)}, 1 \le i \le n-4,$

$f(x_{2i+1}) = l_{4n-6-8(i-1)}, 1 \le i \le n-4$

Next, we claim that the edge labels are distinct. Let,

$$\begin{aligned}
E_1 &= \{ f_1(u_{2i}v_{2i}) : 1 \le i \le n-3\} \\
&= \left\{ \left| f(u_{2i}) - f(v_{2i}) \right| : 1 \le i \le n-3 \right\} \\
&= \left\{ \left| f(u_2) - f(v_2) \right|, \left| f(u_4) - f(v_4) \right| \ldots \left| f(u_{2n-6}) - f(v_{2n-6}) \right| \right\} \\
&= \left\{ \left| l_{4n-2} - l_{4n-1} \right|, \left| l_{4n-10} - l_{4n-9} \right|, \ldots, \left| l_{30-4n} - l_{31-4n} \right| \right\} \\
&= \left\{ l_{4n-3}, l_{4n-11}, \ldots, l_{29-4n} \right\}
\end{aligned}$$

$$\begin{aligned}
E_2 &= \{ f_1(u_{2i+1}v_{2i+1}) : 0 \le i \le n-4\} \\
&= \left\{ \left| f(u_{2i+1}) - f(v_{2i+1}) \right| : 0 \le i \le n-4 \right\} \\
&= \left\{ \left| f(u_1) - f(v_1) \right|, \left| f(u_3) - f(v_3) \right|, \left| f(u_5) - f(v_5) \right| \ldots \left| f(u_{2n-7}) - f(v_{2n-7}) \right| \right\} \\
&= \left\{ \left| l_1 - l_0 \right|, \left| l_{4n-8} - l_{4n-7} \right|, \left| l_{4n-16} - l_{4n-15} \right|, \ldots, \left| l_{32-4n} - l_{33-4n} \right| \right\} \\
&= \left\{ l_1, l_{4n-9}, l_{4n-17}, \ldots, l_{31-4n} \right\}
\end{aligned}$$

$$\begin{aligned}
E_3 &= \{ f_1(v_{2i}w_{2i}) : 1 \le i \le n-3\} \\
&= \left\{ \left| f(v_{2i}) - f(w_{2i}) \right| : 1 \le i \le n-3 \right\} \\
&= \left\{ \left| f(v_2) - f(w_2) \right|, \left| f(v_4) - f(w_4) \right|, \ldots, \left| f(v_{2n-6}) - f(w_{2n-6}) \right| \right\} \\
&= \left\{ \left| l_{4n-1} - l_{4n-3} \right|, \left| l_{4n-9} - l_{4n-11} \right|, \ldots, \left| l_{31-4n} - l_{29-4n} \right| \right\} \\
&= \left\{ l_{4n-2}, l_{4n-10}, \ldots, l_{30-4n} \right\}
\end{aligned}$$

$$\begin{aligned}
E_4 &= \{ f_1(v_{2i+1}w_{2i+1}) : 0 \le i \le n-4\} \\
&= \left\{ \left| f(v_{2i+1}) - f(w_{2i+1}) \right| : 0 \le i \le n-4 \right\} \\
&= \left\{ \left| f(v_1) - f(w_1) \right|, \left| f(v_3) - f(w_3) \right|, \left| f(v_5) - f(w_5) \right|, \ldots, \left| f(v_{2n-7}) - f(w_{2n-7}) \right| \right\} \\
&= \left\{ \left| l_0 - l_2 \right|, \left| l_{4n-15} - l_{4n-13} \right|, \ldots, \left| l_{33-4n} - l_{35-4n} \right| \right\} \\
&= \left\{ l_2, l_{4n-6}, l_{4n-14}, \ldots, l_{34-4n} \right\}
\end{aligned}$$

$$\begin{aligned} E_5 &= \{f_1(w_{2i}x_{2i}) : 1 \le i \le n-3\} \\ &= \{|f(w_{2i}) - f(x_{2i})| : 1 \le i \le n-3\} \\ &= \{|f(w_2) - f(x_2)|, |f(w_4) - f(x_4)|, \ldots, |f(w_{2n-6}) - f(x_{2n-6})|\} \\ &= \{|l_{4n-3} - l_{4n-4}|, |l_{4n-11} - l_{4n-12}|, \ldots, |l_{29-4n} - l_3|\} \\ &= \{l_{4n-5}, l_{4n-13}, \ldots, l_{28-4n}\} \end{aligned}$$

$$\begin{aligned} E_6 &= \{f_1(w_{2i+1}x_{2i+1}) : 0 \le i \le n-4\} \\ &= \{|f(w_{2i+1}) - f(x_{2i+1})| : 0 \le i \le n-4\} \\ &= \{|f(w_1) - f(x_1)|, |f(w_3) - f(x_3)|, |f(w_5) - f(x_5)|, \ldots, |f(w_{2n-7}) - f(x_{2n-7})|\} \\ &= \{|l_2 - l_4|, |l_{4n-5} - l_{4n-6}|, |l_{4n-13} - l_{4n-14}|, \ldots, |l_{35-4n} - l_{34-4n}|\} \\ &= \{l_{4n-21}, l_{4n-15}, \ldots, l_{33-4n}\} \end{aligned}$$

$$\begin{aligned} E_7 &= \{f_1(v_{2i+1}v_{2j}) : 0 \le i \le n-4, 1 \le j \le n-3\} \\ &= \{|f(v_{2i+1}) - f(v_{2j})| : 0 \le i \le n-4, 1 \le j \le n-3\} \\ &= \{|f(v_1) - f(v_2)|, |f(v_3) - f(v_4)|, \ldots, |f(v_{2n-7}) - f(v_{2n-6})|\} \\ &= \{|l_0 - l_{4n-1}|, |l_{4n-7} - l_{4n-9}|, \ldots, |l_{33-4n} - l_{31-4n}|\} \\ &= \{l_{4n-1}, l_{4n-8}, \ldots, l_{32-4n}\} \end{aligned}$$

$$\begin{aligned} E_8 &= \{f_1(w_{2i}w_{2j+1}) : 1 \le i \le n-4, 1 \le j \le n-4\} \\ &= \{|f(w_{2i}) - f(w_{2j+1})| : 1 \le i \le n-4, 1 \le j \le n-4\} \\ &= \{|f(w_2) - f(w_3)|, |f(w_4) - f(w_5)|, \ldots, |f(w_{2n-8}) - f(w_{2n-7})|\} \\ &= \{|l_{4n-3} - l_{4n-5}|, |l_{4n-11} - l_{4n-13}|, \ldots, |l_{37-4n} - l_{35-4n}|\} \\ &= \{l_{4n-4}, l_{4n-12}, \ldots, l_{36-4n}\} \end{aligned}$$

Now, $E = E_1 \cup E_2 \cup \ldots \cup E_8 = \{l_1, l_2, \ldots l_{4n-2}, l_{4n-1}\}$. So, the edges of G receive the distinct labels. Therefore, f is a Lucas graceful labeling. Hence, T_p - tree G with | V (G) | = 4n, (n is even) is a Lucas graceful graph.

Case (ii) when n is odd, the corresponding labeling is defined by,

$f(u_{2i}) = l_{4n-2-8(i-1)}, 1 \le i \le n-4,\ f(v_{2i}) = l_{4n-1-8(i-1)}, 1 \le i \le n-4,$

$f(w_{2i}) = l_{4n-3-8(i-1)}, 1 \le i \le n-4,\ f(x_{2i}) = l_{4n-4-8(i-1)}, 1 \le i \le n-4,$

$f(u_{2n-5}) = l_3,\ f(u_{2i+1}) = l_{4n-8-8(i-1)}, 1 \le i \le n-4$

$f(v_{2i+1}) = l_{4n-7-8(i-1)}, 1 \le i \le n-4,\ f(w_{2i+1}) = l_{4n-5-8(i-1)}, 1 \le i \le n-4,$

$f(x_{2i+1}) = l_{4n-6-8(i-1)}, 1 \le i \le n-4$

Now, we claim that the edge labels are distinct

Let

$$\begin{aligned}
E_1 &= \{f_1(u_{2i}v_{2i}) : 1 \le i \le n-4\} \\
&= \{|f(u_{2i}) - f(v_{2i})| : 1 \le i \le n-4\} \\
&= \{|f(u_2) - f(v_2)|, |f(u_4) - f(v_4)|, \ldots, |f(u_{2n-8}) - f(v_{2n-8})|\} \\
&= \{|l_{4n-2} - l_{4n-1}|, |l_{4n-10} - l_{4n-9}|, \ldots, |l_{38-4n} - l_{37-4n}|\} \\
&= \{l_{4n-3}, l_{4n-11}, \ldots, l_{36-4n}\}
\end{aligned}$$

$$\begin{aligned}
E_2 &= \{f_1(u_{2i+1}v_{2i+1}) : 0 \le i \le n-4\} \\
&= \{|f(u_{2i+1}) - f(v_{2i+1})| : 0 \le i \le n-4\} \\
&= \left\{\begin{matrix}|f(u_1) - f(v_1)|, |f(u_3) - f(v_3)|, |f(u_5) - f(v_5)|, \ldots, \\ |f(u_{2n-7}) - f(v_{2n-7})|\end{matrix}\right\} \\
&= \{|l_1 - l_0|, |l_{4n-8} - l_{4n-7}|, |l_{4n-16} - l_{4n-15}|, \ldots, |l_{32-4n} - l_{31-4n}|\} \\
&= \{l_1, l_{4n-9}, l_{4n-17}, \ldots, l_{30-4n}\}
\end{aligned}$$

$$\begin{aligned}
E_3 &= \{f_1(v_{2i}w_{2i}) : 1 \le i \le n-4\} \\
&= \{|f(v_{2i}) - f(w_{2i})| : 1 \le i \le n-4\} \\
&= \{|f(v_2) - f(w_2)|, |f(v_4) - f(w_4)|, \ldots, |f(v_{2n-8}) - f(w_{2n-8})|\} \\
&= \{|l_{4n-1} - l_{4n-3}|, |l_{4n-9} - l_{4n-11}|, \ldots, |l_{39-4n} - l_{37-4n}|\} \\
&= \{l_{4n-2}, l_{4n-10}, \ldots, l_{38-4n}\}
\end{aligned}$$

$$\begin{aligned}
E_4 &= \{f_1(v_{2i+1}w_{2i+1}) : 0 \le i \le n-3\} \\
&= \{|f(v_{2i+1}) - f(w_{2i+1})| : 0 \le i \le n-3\} \\
&= \left\{\begin{matrix}|f(v_1) - f(w_1)|, |f(v_3) - f(w_3)|, |f(v_5) - f(w_5)|, \ldots, \\ |f(v_{2n-5}) - f(w_{2n-5})|\end{matrix}\right\} \\
&= \{|l_0 - l_2|, |l_{4n-7} - l_{4n-5}|, |l_{4n-15} - l_{4n-13}|, \ldots, |l_{33-4n} - l_{35-4n}|\} \\
&= \{l_2, l_{4n-6}, l_{4n-14}, \ldots, l_{34-4n}\}
\end{aligned}$$

$$E_5 = \{ f_1(w_{2i}x_{2i}) : 1 \le i \le n-4\}$$
$$= \left\{ \left| f(w_{2i}) - f(x_{2i}) \right| : 1 \le i \le n-4 \right\}$$
$$= \left\{ \left| f(w_2) - f(x_2) \right|, \left| f(w_4) - f(x_4) \right|, ..., \left| f(w_{2n-8}) - f(x_{2n-8}) \right| \right\}$$
$$= \left\{ \left| l_{4n-3} - l_{4n-4} \right|, \left| l_{4n-11} - l_{4n-12} \right|, ..., \left| l_{37-4n} - l_{38-4n} \right| \right\}$$
$$= \left\{ l_{4n-5}, l_{4n-13}, ..., l_{39-4n} \right\}$$
$$E_6 = \{ f_1(w_{2i+1}x_{2i+1}) : 0 \le i \le n-4\}$$
$$= \left\{ \left| f(w_{2i+1}) - f(x_{2i+1}) \right| : 0 \le i \le n-4 \right\}$$
$$= \left\{ \begin{matrix} \left| f(w_1) - f(x_1) \right|, \left| f(w_3) - f(x_3) \right|, \left| f(w_5) - f(x_5) \right|, ..., \\ \left| f(w_{2n-7}) - f(x_{2n-7}) \right| \end{matrix} \right\}$$
$$= \left\{ \left| l_2 - l_4 \right|, \left| l_{4n-5} - l_{4n-6} \right|, \left| l_{4n-13} - l_{4n-14} \right|, ..., \left| l_{35-4n} - l_{36-4n} \right| \right\}$$
$$= \left\{ l_3, l_{4n-7}, l_{4n-15}, ..., l_{37-4n} \right\}$$
$$E_7 = \{ f_1(v_{2i+1}v_{2j}) : 0 \le i \le n-5, 1 \le j \le n-4\}$$
$$= \left\{ \left| f(v_{2i+1}) - f(v_{2j}) \right| : 0 \le i \le n-5, 1 \le j \le n-4 \right\}$$
$$= \left\{ \left| f(v_1) - f(v_2) \right|, \left| f(v_3) - f(v_4) \right|, ..., \left| f(v_{2n-9}) - f(v_{2n-8}) \right| \right\}$$
$$= \left\{ \left| l_0 - l_{4n-1} \right|, \left| l_{4n-7} - l_{4n-9} \right|, ..., \left| l_{41-4n} - l_{39-4n} \right| \right\}$$
$$= \left\{ l_{4n-1}, l_{4n-8}, ..., l_{40-4n} \right\}$$
$$E_8 = \{ f_1(w_{2i}w_{2j+1}) : 1 \le i \le n-3, 1 \le j \le n-3\}$$
$$= \left\{ \left| f(w_{2i}) - f(w_{2j+1}) \right| : 1 \le i \le n-3, 1 \le j \le n-3 \right\}$$
$$= \left\{ \left| f(w_2) - f(w_3) \right|, \left| f(w_4) - f(w_5) \right|, ..., \left| f(w_{2n-6}) - f(w_{2n-5}) \right| \right\}$$
$$= \left\{ \left| l_{4n-3} - l_{4n-5} \right|, \left| l_{4n-11} - l_{4n-13} \right|, ..., \left| l_{29-4n} - l_{27-4n} \right| \right\}$$
$$= \left\{ l_{4n-4}, l_{4n-12}, ..., l_{28-4n} \right\}$$

Now, $E = E_1 \cup E_2 \cup ... \cup E_8 = \{l_1, l_2, ..., l_{4n-1}\}$ .So, the edge of G receive the distinct labels. Therefore, f is a Lucas graceful labeling. Hence, T_p- tree with | V (G) | = 4n, (n is odd) is a Lucas graceful graph.

3. **Example**

(i) The T_p-tree admit Lucas graceful labeling, when n is even. When n=6, |V (G) | = 24 using case (i) of theorem 2.2,

Define $f : V(G) \to \{l_0, l_1, ..., l_{23}\}$

The corresponding labeling is defined by,

$f(u_1) = l_1,\ f(v_1) = l_0\ ,\ f(w_1) = l_2,\ f(x_1) = l_4$

$f(u_{2i}) = l_{22-8(i-1)}, 1 \le i \le 3,\ f(v_{2i}) = l_{23-8(i-1)}, 1 \le i \le 3,$

$f(w_{2i}) = l_{21-8(i-1)}, 1 \le i \le 3,\ f(x_{2i}) = l_{20-8(i-1)}, 1 \le i \le 3,$

$f(x_6) = l_3, 1 \le i \le 3,\ f(u_{2i+1}) = l_{16-8(i-1)}, 1 \le i \le 2$

$f(v_{2i+1}) = l_{17-8(i-1)}, 1 \le i \le 2,\ f(w_{2i+1}) = l_{19-8(i-1)}, 1 \le i \le 2,$

$f(x_{2i+1}) = l_{18-8(i-1)}, 1 \le i \le 2$

Next, we claim that the edge labels are distinct. Let

$$\begin{aligned}
E_1 &= \{ f_1(u_{2i}v_{2i}) : 1 \le i \le 3\} \\
&= \{ |f(u_{2i}) - f(v_{2i})| : 1 \le i \le 3\} \\
&= \{ |f(u_2) - f(v_2)|, |f(u_4) - f(v_4)|, |f(u_6) - f(v_6)| \} \\
&= \{ |l_{22} - l_{23}|, |l_{14} - l_{15}|, |l_6 - l_7| \} \\
&= \{ l_{21}, l_{13}, l_5 \}
\end{aligned}$$

$$\begin{aligned}
E_2 &= \{ f_1(u_{2i+1}v_{2i+1}) : 0 \le i \le 2\} \\
&= \{ |f(u_{2i+1}) - f(v_{2i+1})| : 0 \le i \le 2\} \\
&= \{ |f(u_1) - f(v_1)|, |f(u_3) - f(v_3)|, |f(u_5) - f(v_5)| \} \\
&= \{ |l_1 - l_0|, |l_{16} - l_{17}|, |l_8 - l_9| \} \\
&= \{ l_1, l_{15}, l_7 \}
\end{aligned}$$

$$\begin{aligned}
E_3 &= \{ f_1(v_{2i}w_{2i}) : 1 \le i \le 3\} \\
&= \{ |f(v_{2i}) - f(w_{2i})| : 1 \le i \le 3\} \\
&= \{ |f(v_2) - f(w_2)|, |f(v_4) - f(w_4)| |f(v_6) - f(w_6)| \} \\
&= \{ |l_{23} - l_{21}|, |l_{15} - l_{13}|, |l_7 - l_5| \} \\
&= \{ l_{22}, l_{14}, l_6 \}
\end{aligned}$$

$$\begin{aligned}
E_4 &= \{ f_1(v_{2i+1}w_{2i+1}) : 0 \le i \le 2\} \\
&= \{ |f(v_{2i+1}) - f(w_{2i+1})| : 0 \le i \le 2\} \\
&= \{ |f(v_1) - f(w_1)|, |f(v_3) - f(w_3)|, |f(v_5) - f(w_5)| \} \\
&= \{ |l_0 - l_2|, |l_{17} - l_{19}|, |l_9 - l_{11}| \} \\
&= \{ l_2, l_{18}, l_{10} \}
\end{aligned}$$

$$
\begin{aligned}
E_5 &= \{ f_1(w_{2i}x_{2i}) : 1 \le i \le 3\} \\
&= \left\{ \left| f(w_{2i}) - f(x_{2i}) \right| : 1 \le i \le 3 \right\} \\
&= \left\{ \left| f(w_2) - f(x_2) \right|, \left| f(w_4) - f(x_4) \right|, \left| f(w_6) - f(x_6) \right| \right\} \\
&= \left\{ \left| l_{21} - l_{20} \right|, \left| l_{13} - l_{12} \right|, \left| l_5 - l_3 \right| \right\} \\
&= \left\{ l_{19}, l_{11}, l_4 \right\}
\end{aligned}
$$

$$
\begin{aligned}
E_6 &= \{ f_1(w_{2i+1}x_{2i+1}) : 0 \le i \le 2\} \\
&= \left\{ \left| f(w_{2i+1}) - f(x_{2i+1}) \right| : 0 \le i \le 2 \right\} \\
&= \left\{ \left| f(w_1) - f(x_1) \right|, \left| f(w_3) - f(x_3) \right|, \left| f(w_5) - f(x_5) \right| \right\} \\
&= \left\{ \left| l_2 - l_4 \right|, \left| l_{19} - l_{18} \right|, \left| l_{11} - l_{10} \right| \right\} = \left\{ l_3, l_{17}, l_9 \right\}
\end{aligned}
$$

$$
\begin{aligned}
E_7 &= \{ f_1(v_{2i+1}v_{2j}) : 0 \le i \le 2, 1 \le j \le 3\} \\
&= \left\{ \left| f(v_{2i+1}) - f(v_{2j}) \right| : 0 \le i \le 2, 1 \le j \le 3 \right\} \\
&= \left\{ \left| f(v_1) - f(v_2) \right|, \left| f(v_3) - f(v_4) \right|, \left| f(v_5) - f(v_6) \right| \right\} \\
&= \left\{ \left| l_0 - l_{23} \right|, \left| l_{17} - l_{15} \right|, \left| l_9 - l_7 \right| \right\} \\
&= \left\{ l_{23}, l_{16}, l_8 \right\}
\end{aligned}
$$

$$
\begin{aligned}
E_8 &= \{ f_1(w_{2i}w_{2j+1}) : 1 \le i \le 2, 1 \le j \le 2\} \\
&= \left\{ \left| f(w_{2i}) - f(w_{2j+1}) \right| : 1 \le i \le 2, 1 \le j \le 2 \right\} \\
&= \left\{ \left| f(w_2) - f(w_3) \right|, \left| f(w_4) - f(w_5) \right| \right\} \\
&= \left\{ \left| l_{21} - l_{19} \right|, \left| l_{13} - l_{11} \right| \right\} \\
&= \left\{ l_{20}, l_{12} \right\}
\end{aligned}
$$

Now, $E = E_1 \cup E_2 \cup ... \cup E_8 = \{l_1, l_2, l_3, ..., l_{23}\}$. So, the edges of G receive the distinct labels. Therefore, f is a Lucas graceful labeling. Hence, T_p - tree G with | V (G) | = 4n, (n is even) admits a Lucas graceful labeling which is shown in Fig.3.

(ii) The T_p-tree admit Lucas graceful labeling, when n is odd. When n=7, | V (G) | = 28 using case (ii) of theorem 2.2, Define

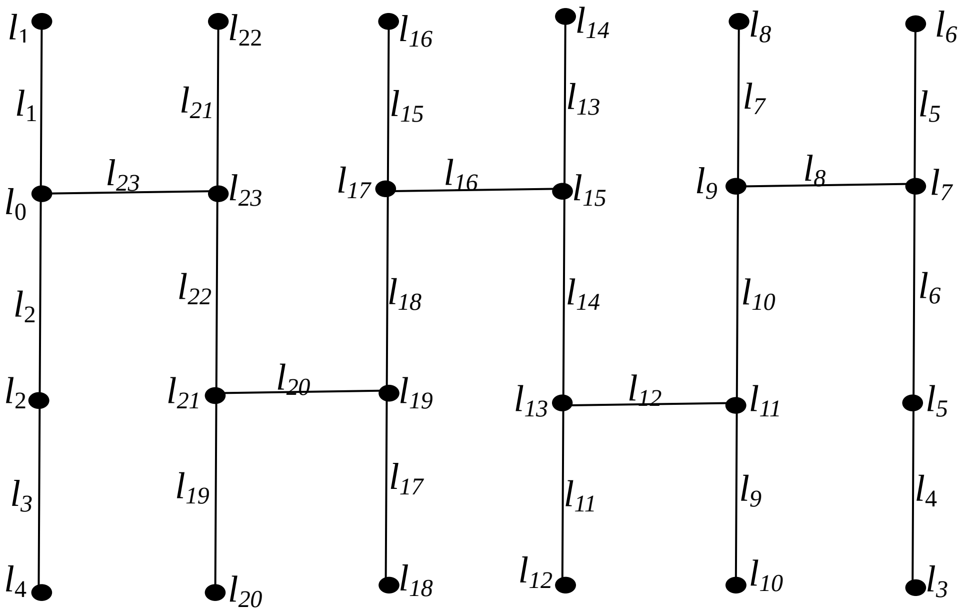


Fig. 3

$f : V(G) \to \{l_0, l_1, ...l_{27}\}$

The corresponding labeling is defined by,

$f(u_1) = l_1,\ f(v_1) = l_0\ ,\ f(w_1) = l_2,\ f(x_1) = l_4$

$f(u_{2i}) = l_{26-8(i-1)}, 1 \le i \le 3,\ f(v_{2i}) = l_{27-8(i-1)}, 1 \le i \le 3,$

$f(w_{2i}) = l_{25-8(i-1)}, 1 \le i \le 3,\ f(x_{2i}) = l_{24-8(i-1)}, 1 \le i \le 3,$

$f(u_{2n-5}) = l_3,\ f(u_{2i+1}) = l_{20-8(i-1)}, 1 \le i \le 3$

$f(v_{2i+1}) = l_{21-8(i-1)}, 1 \le i \le 3,\ f(w_{2i+1}) = l_{23-8(i-1)}, 1 \le i \le 3,$

$f(x_{2i+1}) = l_{22-8(i-1)}, 1 \le i \le 3$

Now, we claim that the edge labels are distinct, Let,

$$
\begin{aligned}
E_1 &= \{f_1(u_{2i}v_{2i}) : 1 \le i \le 3\} \\
&= \{|f(u_{2i}) - f(v_{2i})| : 1 \le i \le 3\} \\
&= \{|f(u_2) - f(v_2)|, |f(u_4) - f(v_4)|, |f(u_6) - f(v_6)|\} \\
&= \{|l_{26} - l_{27}|, |l_{18} - l_{19}|, |l_{10} - l_{11}|\} \\
&= \{l_{25}, l_{17}, l_9\}
\end{aligned}
$$

$$
\begin{aligned}
E_2 &= \{ f_1(u_{2i+1}v_{2i+1}) : 0 \le i \le 3\} \\
&= \left\{ \left| f(u_{2i+1}) - f(v_{2i+1}) \right| : 0 \le i \le 3 \right\} \\
&= \left\{ \left| f(u_1) - f(v_1) \right|, \left| f(u_3) - f(v_3) \right|, \left| f(u_5) - f(v_5) \right|, \left| f(u_7) - f(v_7) \right| \right\} \\
&= \left\{ \left| l_1 - l_0 \right|, \left| l_{20} - l_{21} \right|, \left| l_{12} - l_{13} \right|, \left| l_3 - l_5 \right| \right\} \\
&= \left\{ l_1, l_{19}, l_{11}, l_4 \right\}
\end{aligned}
$$

$$
\begin{aligned}
E_3 &= \{ f_1(v_{2i}w_{2i}) : 1 \le i \le 3\} \\
&= \left\{ \left| f(v_{2i}) - f(w_{2i}) \right| : 1 \le i \le 3 \right\} \\
&= \left\{ \left| f(v_2) - f(w_2) \right|, \left| f(v_4) - f(w_4) \right|, \left| f(v_6) - f(w_6) \right| \right\} \\
&= \left\{ \left| l_{27} - l_{25} \right|, \left| l_{19} - l_{17} \right|, \left| l_{11} - l_9 \right| \right\} \\
&= \left\{ l_{26}, l_{18}, l_{10} \right\}
\end{aligned}
$$

$$
\begin{aligned}
E_4 &= \{ f_1(v_{2i+1}w_{2i+1}) : 0 \le i \le 3\} \\
&= \left\{ \left| f(v_{2i+1}) - f(w_{2i+1}) \right| : 0 \le i \le 3 \right\} \\
&= \left\{ \left| f(v_1) - f(w_1) \right|, \left| f(v_3) - f(w_3) \right|, \left| f(v_5) - f(w_5) \right|, \left| f(v_7) - f(w_7) \right| \right\} \\
&= \left\{ \left| l_0 - l_2 \right|, \left| l_{21} - l_{23} \right|, \left| l_{13} - l_{15} \right|, \left| l_5 - l_7 \right| \right\} \\
&= \left\{ l_2, l_{22}, l_{14}, l_6 \right\}
\end{aligned}
$$

$$
\begin{aligned}
E_5 &= \{ f_1(w_{2i}x_{2i}) : 1 \le i \le 3\} \\
&= \left\{ \left| f(w_{2i}) - f(x_{2i}) \right| : 1 \le i \le 3 \right\} \\
&= \left\{ \left| f(w_2) - f(x_2) \right|, \left| f(w_4) - f(x_4) \right|, \left| f(w_6) - f(x_6) \right| \right\} \\
&= \left\{ \left| l_{25} - l_{24} \right|, \left| l_{17} - l_{16} \right|, \left| l_9 - l_8 \right| \right\} \\
&= \left\{ l_{23}, l_{15}, l_7 \right\}
\end{aligned}
$$

$$
\begin{aligned}
E_6 &= \{ f_1(w_{2i+1}x_{2i+1}) : 0 \le i \le 3\} \\
&= \left\{ \left| f(w_{2i+1}) - f(x_{2i+1}) \right| : 0 \le i \le 3 \right\}
\end{aligned}
$$

$= \{|f(w_1)-f(x_1)|,|f(w_3)-f(x_3)|,|f(w_5)-f(x_5)|,|f(w_7)-f(x_7)|\}$

$= \{|l_2-l_4|,|l_{23}-l_{22}|,|l_{15}-l_{14}|,|l_7-l_6|\}$

$= \{l_3,l_{21},l_{13},l_5\}$

$E_7 = \{f_1(v_{2i+1}v_{2j}) : 0\le i\le 2, 1\le j\le 3\}$

$= \{|f(v_{2i+1})-f(v_{2j})| : 0\le i\le 2, 1\le j\le 3\}$

$= \{|f(v_1)-f(v_2)|,|f(v_3)-f(v_4)|,|f(v_5)-f(v_6)|\}$

$= \{|l_0-l_{27}|,|l_{21}-l_{19}|,|l_{13}-l_{11}|\}$

$= \{l_{27},l_{20},l_{12}\}$

$E_8 = \{f_1(w_{2i}w_{2j+1}) : 1\le i\le 3, 1\le j\le 3\}$

$= \{|f(w_{2i})-f(w_{2j+1})| : 1\le i\le 3, 1\le j\le 3\}$

$= \{|f(w_2)-f(w_3)|,|f(w_4)-f(w_5)|,|f(w_6)-f(w_7)|\}$

$= \{|l_{25}-l_{23}|,|l_{17}-l_{15}|,|l_9-l_7|\}$

$= \{l_{24},l_{16},l_8\}$

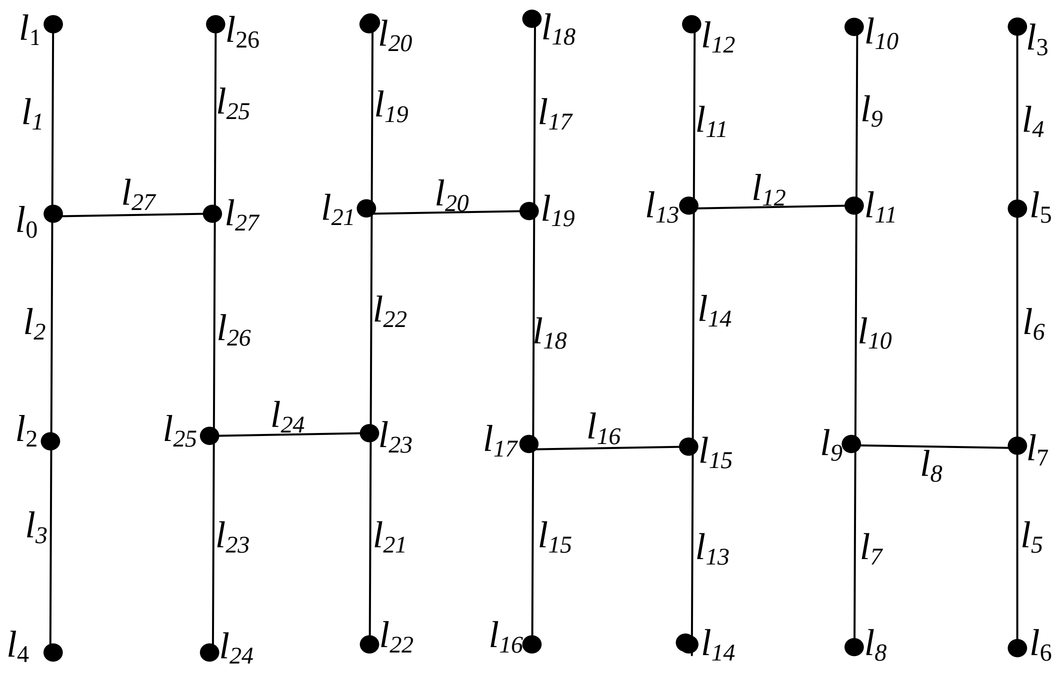


Fig. 4

Now, $E = E_1 \cup E_2 \cup ... \cup E_8 = \{l_1,l_2,l_3,...,l_{27}\}$. So, the edge of G receive the distinct labels. Therefore, f is a Lucas graceful labeling. Hence, T_p- tree with | V (G) | = 4n, (n is odd) admits a Lucas graceful labeling which is shown inFig.4.

References

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